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Λ-symmetry and background independence of noncommutative gauge theory on $\mathbb{R}^n$

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Abstract: Background independence of noncommutative Yang-Mills theory on $\mathbb{R}^n$ is discussed. The quantity $\theta\hat{F}\theta - \theta$ is found to be background dependent at subleading order, and it becomes background independent only when the ordinary gauge field strength F is constant. It is shown that, at small values of B, the noncommutative Dirac-Born-Infeld action possesses Λ-symmetry at least to subleading order in θ if F damps fast enough at infinity.

Keywords: D-branes, Space-Time Symmetries, Gauge Symmetry.

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Contents

1. Introduction

Recently, Seiberg and Witten observed that ordinary and noncommutative gauge fields can be induced by the same $2D$ σ-model action regularized in different ways [1]. Then they argued that ordinary gauge fields should be equivalent to noncommutative gauge theory, which, in a certain limit, acts as the low energy effective theory of open stings. Furthermore, they conjectured that the noncommutative effective lagrangean takes the same form as the ordinary Dirac-Born-Infeld (DBI) action for the D-brane, except that the product of functions is replaced by the star product and Lorentz indices are contracted by the open-string metric G instead of the closed-string metric g. The explicit transformation between the ordinary gauge field A and the noncommutative one $\hat{A}$ was given up to order θ [1]. Even at this order there is an ambiguity in the general transformation between A and $\hat{A}$, which, however, can be removed by a field redefinition of the noncommutative gauge field [2].

The classical σ-model action in a B-field background has two abelian symmetries: One is the U(1) gauge symmetry $A \to A + d\,\lambda$, and the other is the "Λ-symmetry" $A \to A - \Lambda$, $B \to B + d\Lambda$, which keeps $B + F$ invariant. In [3] it was argued that the U(1) gauge symmetry is enhanced to a U(N) when N D -branes coincide, while the Λ-symmetry is believed to act on the U(1) part of the U(N) symmetry. In the ordinary DBI lagrangean, the Λ-symmetry is manifest. So it is natural to ask, whether the noncommutative gauge theory has Λ-symmetry or not if its ordinary counterpart has this symmetry.

In ref. [1] background independence of noncommutative Yang-Mills theory on $\mathbb{R}^n$ was considered. In ordinary Yang-Mills theory, the gauge-invariant combination of B and F is $M = B + F$. The same gauge invariant field M can be split in different ways as $B+F$ or $B'+F'$. In [1], it was shown that the noncommutative Yang-Mills theory on $\mathbb{R}^n$ is invariant under those different splitting, which was called background independence of the noncommutative Yang-Mills theory. Of course this is nothing but the Λ-symmetry in the context of the noncommutative Yang-Mills theory. What Seiberg and Witten did in [1] is that they analysed the Λ-symmetry in the framework of ordinary Yang-Mills theory on noncommutative spacetime. Since the ordinary gauge theory on noncommutative spacetime is supposed to be equivalent to noncommutative gauge theory on commutative (i.e. ordinary) spacetime [4], it is interesting to see how Λ-symmetry is realized in noncommutative Yang-Mills theory on ordinary spacetime.

In the present paper we discuss the background independence of noncommutative Yang-Mills theory on $\mathbb{R}^n$ from the point of view of the noncommutative gauge theory on ordinary spacetime. We first give the explicit expression for the transformation between ordinary $U(1)$ gauge fields and noncommutative ones up to second order in θ,[1] which can be obtained from the differential equation given in [1]. We check the Λ-symmetry of the noncommutative Yang-Mills theory on $\mathbb{R}^n$ up to subleading order. We find that for the noncommutative Yang-Mills field on ordinary spacetime, the quantity $Q=\theta\hat{F}\theta-\theta$ is background dependent for general F, and only when F is constant we have $Q=-\left(B+F\right)^{-1}$ which is background independent. The measure $d^nx\sqrt{G}/g_{YM}^2$ varies with θ, but the noncommutative Yang-Mills action is invariant under Λ-symmetry if the ordinary field strength F damps fast enough at infinity. We further show that the noncommutative DBI action has Λ-symmetry in the case of small B (i.e. small θ) up to subleading order. All of this indicates that noncommutative gauge theories, which act as low energy effective theories of open strings, indeed possess Λ-symmetry.

The paper is organized as follows. In the next section, we first give the expression for the noncommutative field strength $\hat{F}(\hat{A})$ in terms of the ordinary A and F up to order θ^2. We examine the quantity Q and the measure $d^nx\sqrt{G}/g_{YM}^2$ in the context of noncommutative gauge theory on ordinary spacetime. We show that the noncommutative Yang-Mills action is background independent. In section 3, we check Λ-symmetry for the noncommutative DBI action for small values of B. In section 4, we present our summary and discussion.

2. Background independence of noncommutative Yang-Mills theory on $\mathbb{R}^n$

Demanding that ordinary gauge fields that are gauge-equivalent are mapped to noncommutative gauge fields that are likewise gauge-equivalent, the transformation be-

[1]The expression for $\hat{A}$ in terms of A up to order θ^2 was given in [5].

tween the noncommutative gauge field $\hat{A}$ and the ordinary A can be described by [1]

$$\hat{A}\left(A\right)+\delta_{\hat{\lambda}}\,\hat{A}\left(A\right)=\hat{A}\left(A+\delta_{\lambda}A\right), \tag{2.1}$$

with infinitesimal λ and $\hat{\lambda}(\lambda, A)$.

The equation (2.1) can be solved by expanding it in powers of the noncommutative parameter θ. The differential equations, which describe how $\hat{A}\left(\theta\right)$ and $\hat{\lambda}(\theta)$ should change when θ varies, are given by [1]

$$\begin{aligned}
\delta\hat{A}_i\left(\theta\right) &= -\frac{1}{4}\delta\theta^{kl}\left[\hat{A}_k*\left(\partial_l\hat{A}_i+\hat{F}_{li}\right)+\left(\partial_l\hat{A}_i+\hat{F}_{li}\right)*\hat{A}_k\right] && (2.2)\\
\delta\hat{\lambda}\left(\theta\right) &= \frac{1}{4}\delta\theta^{kl}\left(\partial_k\hat{\lambda}*\hat{A}_l+\hat{A}_l*\partial_k\,\hat{\lambda}\right) && (2.3)\\
\delta\hat{F}_{ij}(\theta) &= \frac{1}{4}\delta\theta^{kl}\Big[2\hat{F}_{ik}*\hat{F}_{jl}+2\hat{F}_{jl}*\hat{F}_{ik}-\hat{A}_k*\left(\hat{D}_l\,\hat{F}_{ij}+\partial_l\hat{F}_{ij}\right)- \\
&\qquad -\left(\hat{D}_l\,\hat{F}_{ij}+\partial_l\hat{F}_{ij}\right)*\hat{A}_k\Big]\,. && (2.4)
\end{aligned}$$

Replacing $\hat{A}$ by A on the right side of the above equations, we get the solutions up to order θ, and for $U\left(1\right)$ gauge fields we have.[2]

$$\begin{aligned}
\hat{A}_i^{(1)} &= A_i-\frac{1}{2}\theta^{kl}A_k\left(\partial_lA_i+F_{li}\right)+\mathcal{O}\left(\theta^2\right)\\
\hat{F}_{ij}^{(1)} &= F_{ij}-\theta^{kl}\left(F_{ik}\,F_{lj}+A_k\partial_lF_{ij}\right)+\mathcal{O}\left(\theta^2\right). && (2.5)
\end{aligned}$$

Inserting $\hat{A}^{(1)}$ and $\hat{F}^{(1)}$ into the right hand side of equations (2.2) and (2.4) we obtain $\hat{A}$ and $\hat{F}$ up to order θ^2,

$$\begin{aligned}
\hat{A}_i &= A_i-\frac{1}{2}\theta^{kl}A_k\left(\partial_lA_i+F_{li}\right)+\\
&\quad +\frac{1}{2}\theta^{kl}\theta^{mn}\left\{A_k\left[\partial_lA_m\partial_nA_i-\left(\partial_lF_{mi}\right)A_n+F_{lm}F_{ni}\right]\right\}+\mathcal{O}\left(\theta^3\right), && (2.6)
\end{aligned}$$

$$\hat{F} = F-F\theta F-\theta^{kl}A_k\partial_lF+\mathcal{T}_{\theta^2}+\mathcal{O}\left(\theta^3\right), \tag{2.7}$$

with

$$\begin{aligned}
\mathcal{T}_{\theta^2} &= F\theta F\theta F+\frac{1}{2}\,A_k\theta^{kl}\left(\partial_lA_m+F_{lm}\right)\theta^{mn}\partial_nF+\\
&\quad +\theta^{kl}A_k\partial_l(F\theta F)+\frac{1}{2}\,\theta^{kl}\theta^{mn}A_kA_m\partial_l\partial_nF\,. && (2.8)
\end{aligned}$$

To obtain eqs. (2.6)–(2.8), we solved eqs. (2.2) and (2.4) perturbatively, and we treat the ordinary F as θ independent, which is different from the following discussion about Λ-transformations, where F will vary with θ.

[2] We are interested in $\hat{A}$, especially in $\hat{F}$, so we do not calculate $\hat{\lambda}$.

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Now let us consider noncommutative Yang-Mills theory on $\mathbb{R}^n$. The action is [1]

$$\hat{L} = \frac{1}{g_{YM}^2} \int d^n x \sqrt{G} G^{ik} G^{jl} \left(\hat{F}_{ij} - \theta_{ij}^{-1} \right) * \left(\hat{F}_{kl} - \theta_{kl}^{-1} \right), \tag{2.9}$$

with G and θ defined by

$$\theta = B^{-1}, \qquad G = -B g^{-1} B, \tag{2.10}$$

where we have put $2\pi\alpha' = 1$ for simplicity. Ignoring total derivatives and constant terms we rewrite the action (2.9) as

$$\hat{L} = -\frac{1}{g_{YM}^2} \int d^n x \sqrt{G}\, \mathrm{Tr} \left(\hat{Q} g \hat{Q} g \right), \tag{2.11}$$

with

$$\hat{Q} = \theta \hat{F} \theta. \tag{2.12}$$

Hence it is sufficient to show that the action (2.11) is background independent.

Consider the Λ-transformation, defined by

$$F \to F - \delta B = F + \theta^{-1} \delta\theta \theta^{-1}, \tag{2.13}$$

which keeps $F + B$ invariant. From (2.10) we know that under a Λ-transformation θ changes as $\theta \to \theta + \delta\theta$, while the ordinary field strength F transforms as $F \to F + \theta^{-1}\delta\theta\theta^{-1}$. Here we should point out that, even for small θ, we will assume $\delta\theta \ll \theta$, and of course we only need to keep terms linear in $\delta\theta$.

Under a Λ-transformation $\hat{F}$ changes as

$$\begin{aligned} \hat{F}(\theta + \delta\theta) = {} & F + \theta^{-1}\delta\theta\theta^{-1} - \left(F + \theta^{-1}\delta\theta\theta^{-1}\right)(\theta + \delta\theta)\left(F + \theta^{-1}\delta\theta\theta^{-1}\right) - \\ & - \left[A_k - \frac{1}{2}\left(\theta^{-1}\delta\theta\theta^{-1}\right)_{ks} x^s \right] (\theta + \delta\theta)^{kl} \partial_l F + \\ & + \mathcal{T}_{(\theta+\delta\theta)^2} + \mathcal{O}\left[(\theta + \delta\theta)^3\right]. \end{aligned} \tag{2.14}$$

Collecting the terms linear in $\delta\theta$ we thus obtain

$$\begin{aligned} \delta\hat{F}(\theta) = {} & \theta^{-1}\delta\theta\theta^{-1} - \theta^{-1}\delta\theta F - F\delta\theta\theta^{-1} - F\delta\theta F - \\ & - \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^l{}_s x^s \partial_l F - A_k \delta\theta^{kl} \partial_l F + H_{\delta\theta} + \text{higher order terms}, \end{aligned} \tag{2.15}$$

where $H_{\delta\theta}$ represents the $\delta\theta$ order terms coming from $\mathcal{T}_{(\theta+\delta\theta)^2} - \mathcal{T}_{\theta^2}$, which is obtained by replacing one of the F's by $\theta^{-1}\delta\theta\theta^{-1}$ or an A_i by $-\frac{1}{2}\left(\theta^{-1}\delta\theta\theta^{-1}\right)_{ij} x^j$ in $\mathcal{T}_{\theta^2}$. It has the form

$$\begin{aligned} H_{\delta\theta} = {} & \theta^{-1}\delta\theta F\theta F + F\delta\theta F + F\theta F\delta\theta\,\theta^{-1} + \\ & + \frac{1}{4}\left(\delta\theta\,\theta^{-1}\right)^l{}_k x^k \left(\partial_l A_m + F_{lm}\right)\theta^{mn}\partial_n F + \\ & + \frac{3}{4} A_k \delta\theta^{kl} \partial_l F + \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^l{}_k x^k \partial_l (F\theta F) + \\ & + A_k \theta^{kl} \partial_l \left(\theta^{-1}\delta\theta F + F\delta\theta\theta^{-1}\right) + \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^l{}_k x^k \theta^{mn} A_m \partial_l \partial_n F. \end{aligned} \tag{2.16}$$

Inserting (2.16) into (2.15), then $\delta\hat{F}$ can be written as

$$\begin{aligned}\delta\hat{F}(\theta) = {} & \theta^{-1}\delta\theta\theta^{-1} - \theta^{-1}\delta\theta F - F\delta\theta\theta^{-1} - \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^{l}{}_{k}\, x^{k}\partial_{l}F - \\ & -\frac{1}{4}A_{k}\delta\theta^{kl}\partial_{l}F + \theta^{-1}\delta\theta F\theta F + F\theta F\delta\theta\theta^{-1} + \\ & +\frac{1}{4}\left(\delta\theta\theta^{-1}\right)^{l}{}_{k}\, x^{k}\left(\partial_{l}A_{m} + F_{lm}\right)\theta^{mn}\partial_{n}F + \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^{l}{}_{k}\, x^{k}\partial_{l}\left(F\theta F\right) + \\ & + A_{k}\theta^{kl}\partial_{l}\left(\theta^{-1}\delta\theta F + F\delta\theta\theta^{-1}\right) + \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^{l}{}_{k}\, x^{k}\theta^{mn}A_{m}\partial_{l}\partial_{n}F\,, \end{aligned} \tag{2.17}$$

where the leading order term is $\theta^{-1}\delta\theta\theta^{-1}\left(\theta^{-2}\delta\theta\right)$, the subleading order term is $\theta^{-1}\delta\theta$ and the third order term is $\delta\theta$. Since the leading order term $\theta^{-2}\delta\theta$ gives a trivial contribution to the variation of the noncommutative Yang-Mills lagrangean, the subleading order term $\theta^{-1}\delta\theta$ is enhanced to the nontrivial leading order and the $\delta\theta$ terms are lifted to subleading order.

Before proceeding, we would like to compare $\delta\hat{F}(\theta)$ defined in (2.17) with the variation $\delta\hat{F}(\theta)$ in (2.4), which describes the variation of $\hat{F}$ under a change of the noncommutative parameter θ with the ordinary field strength F kept fixed. Under a Λ-transformation, on the other hand, θ changes, and the ordinary field strength varies simultaneously to keep $B+F$ invariant. In (2.17), when we vary θ (induced by changing B) we keep g_s and g fixed, and we are sticking with point-splitting regularization. In [1], Seiberg and Witten argued that one can use a suitable regularization which somehow interpolates between Pauli-Villars and point-splitting, then one can vary θ while holding g_s, g and B fixed. Thus $\delta\hat{F}$ defined in (2.17) is conceptually different from (2.4).

The variation of the quantity $\hat{Q}$ defined in (2.12) is

$$\begin{aligned}\delta\hat{Q} = {} & \delta\theta - \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^{l}{}_{k}\, x^{k}\partial_{l}\left(\theta F\theta\right) + \\ & + \Big\{ -\frac{1}{4}A_{k}\delta\theta^{kl}\partial_{l}\left(\theta F\theta\right) + \frac{1}{4}\left(\delta\theta\theta^{-1}\right)^{l}{}_{k}\, x^{k}\left(\partial_{l}A_{m} + F_{lm}\right)\theta^{mn}\partial_{n}\left(\theta F\theta\right) + \\ & + \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^{l}{}_{k}\, x^{k}\partial_{l}\left(\theta F\theta F\theta\right) + \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^{l}{}_{k}\, x^{k}A_{m}\theta^{mn}\partial_{l}\partial_{n}\left(\theta F\theta\right)\Big\}\,, \end{aligned} \tag{2.18}$$

where the first term is of order $\delta\theta$, the second is of order $\theta\delta\theta$, and the terms in the bracket are of the order of $\theta^{2}\delta\theta$.

The noncommutative Yang-Mills action (2.11) can be recast into

$$\hat{L} = \frac{1}{g_s}\int d^{n}x\ \det{}^{-1/2}\theta\ \mathrm{Tr}\left(\hat{Q}g\hat{Q}g\right), \tag{2.19}$$

where we have exploited the θ dependence of g_{YM}^{2} [1] and omitted an irrelevant numerical factor. Since we study the noncommutative Yang-Mills theory on ordinary spacetime $\mathbb{R}^n$, the measure $d^n x$ is invariant under the change of θ. Then the variation of the noncommutative Yang-Mills action is

$$\delta \hat{L} = \frac{2}{g_s} \int d^n x \, \det{}^{-1/2}\theta \, \delta S + \text{higher order terms}\,, \tag{2.20}$$

with

$$\delta S = \delta S_1 + \delta S_2 = -\frac{1}{4}\left(\operatorname{Tr}\theta^{-1}\delta\theta\right) \operatorname{Tr}\left(\hat{Q} g \hat{Q} g\right) + \operatorname{Tr}\left[\left(\delta\hat{Q}\right) g \hat{Q} g\right], \tag{2.21}$$

where δS_1 and δS_2 represent the leading and the subleading order, respectively.

Since $\hat{F}$ is a total derivative for U(1) gauge fields, $\delta\theta$ in δQ can be thrown away up to a total derivative. The leading order term in δS is of order $\theta^3\delta\theta$, and the subleading order is $\theta^4\delta\theta$.

At first, let us examine the leading order term δS_1 with

$$\begin{aligned}
\delta S_1 &= -\frac{1}{4}\operatorname{Tr}\theta^{-1}\delta\theta \, \operatorname{Tr}\left(\theta F \theta g \theta F \theta g\right) - \\
&\quad -\frac{1}{2}\operatorname{Tr}\left\{\left(\delta\theta\theta^{-1}\right)^l{}_k \, x^k \left[\partial_l\left(\theta F \theta\right)\right] g \theta F \theta g\right\} \\
&= -\frac{1}{4}\operatorname{Tr}\left(\theta^{-1}\delta\theta\right) \operatorname{Tr}\left(\theta F \theta g \theta F \theta g\right) - \\
&\quad -\frac{1}{4}\operatorname{Tr}\left\{\left(\delta\theta\theta^{-1}\right)^l{}_k \, x^k \left[\partial_l\left(\theta F \theta g\right)^2\right]\right\} \\
&= \text{total derivative}\,,
\end{aligned} \tag{2.22}$$

where from the second step to the last one we have integrated by parts. Eq. (2.22) indicates that the noncommutative Yang-Mills action is background independent at leading order if the ordinary gauge field A damps fast enough at the infinity.

Next we consider the subleading term δS_2, which is of order $\theta^4\delta\theta$,

$$\begin{aligned}
\delta S_2 = &-\frac{1}{4}\operatorname{Tr}\theta^{-1}\delta\theta \times 2\operatorname{Tr}\left[\theta F \theta g \theta\left(-F\theta F - A_k\theta^{kl}\partial_l F\right)\theta g\right] + \\
&+\operatorname{Tr}\left[-\frac{1}{2}\left(\delta\theta\theta^{-1}\right)^l{}_k \, x^k \partial_l\left(\theta F \theta\right) g \theta\left(-F\theta F - A_m\theta^{mn}\partial_n F\right)\theta g\right] + \\
&+\operatorname{Tr}\left\{\left[-\frac{1}{4}A_k\delta\theta^{kl}\partial_l\left(\theta F \theta\right) + \frac{1}{4}\left(\delta\theta\theta^{-1}\right)^l{}_k \, x^k\left(\partial_l A_m + F_{lm}\right)\theta^{mn}\partial_n\left(\theta F \theta\right) + \right.\right. \\
&\qquad \left.\left. + \frac{1}{2}(\delta\theta\theta^{-1})^l{}_k \, x^k \partial_l(\theta F \theta F \theta) + \frac{1}{2}(\delta\theta\theta^{-1})^l{}_k \, x^k A_m\theta^{mn}\partial_l\partial_n(\theta F \theta)\right] g\theta F \theta g\right\}.
\end{aligned} \tag{2.23}$$

Putting similar terms together we have

$$\begin{aligned}
\delta S_2 = &\frac{1}{2}\left(\delta\theta\theta^{-1}\right)^l{}_k \, x^k A_m\theta^{mn}\partial_l \operatorname{Tr}\left[\partial_n\left(\theta F \theta g\right)\theta F \theta g\right] - \\
&-\frac{1}{8}\left(\operatorname{Tr}\theta^{-1}\delta\theta\right)\left(\operatorname{Tr}\theta F\right)\operatorname{Tr}\left(\theta F \theta g \theta F \theta g\right) + \frac{1}{16}\left(\operatorname{Tr}\delta\theta F\right)\operatorname{Tr}\left(\theta F \theta g \theta F \theta g\right) + \\
&+\frac{1}{4}\left(\delta\theta\theta^{-1}\right)^l_k \, x^k\left(\partial_m A_l\right)\theta^{mn}\partial_n\left[\operatorname{Tr}\left(\theta F \theta g \theta F \theta g\right)\right] - \\
&-\frac{1}{8}\left(\delta\theta\theta^{-1}\right)^l_k \, x^k\left(\partial_m A_l\right)\theta^{mn}\partial_n\left[\operatorname{Tr}\left(\theta F \theta g \theta F \theta g\right)\right].
\end{aligned} \tag{2.24}$$

After integration by parts, we get

$$\delta S_2 = \text{total derivative}\,, \tag{2.25}$$

which shows that under Λ-transformation the noncommutative Yang-Mills lagrangean is invariant up to a total derivative also at the subleading order.

To compare our result with those in ref. [1], we define the quantity Q by

$$Q = \hat{Q} - \theta = \theta \hat{F} \theta - \theta\,. \tag{2.26}$$

Under a Λ-transformation, the variation of Q can be obtained from (2.18) and is given by

$$\begin{aligned}\delta Q = & -\frac{1}{2}\left(\delta\theta\theta^{-1}\right)^l{}_k\, x^k \partial_l\left(\theta F \theta\right) - \frac{1}{4} A_k \delta\theta^{kl}\partial_l\left(\theta F\theta\right) + \\ & + \frac{1}{4}\left(\delta\theta\theta^{-1}\right)^l{}_k\, x^k\left(\partial_l A_m + F_{lm}\right)\theta^{mn}\partial_n\left(\theta F\theta\right) + \\ & + \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^l{}_k\, x^k\partial_l\left(\theta F\theta F\theta\right) + \frac{1}{2}\left(\delta\theta\theta^{-1}\right)^l{}_k\, x^k A_m\theta^{mn}\partial_l\partial_n\left(\theta F\theta\right) + \\ & + \text{higher order terms}\,,\end{aligned} \tag{2.27}$$

which shows that the quantity Q is not invariant under Λ-transformation for general ordinary field A up to the subleading order term, but when F is constant, we see $\delta Q = 0 +$ higher order terms, and in this case Q can be expressed as [1]

$$Q = -\frac{1}{B+F}\,, \tag{2.28}$$

where F is constant.

What we have learned from the above discussion is that when we consider the noncommutative Yang-Mills theory on ordinary spacetime, the quantity Q generally is not background independent, and the measure

$$\frac{d^n x\sqrt{G}}{g_{YM}^2} \tag{2.29}$$

should also be background dependent. However, the total noncommutative Yang-Mills action is background independent if the ordinary gauge field damps fast enough at infinity. Because of the equivalence between ordinary gauge theory on noncommutative spacetime and noncommutative gauge theory on commutative (i.e. ordinary) spacetime [4], we believe that in the context of ordinary gauge theory on noncommutative spacetime, the quantity Q and the measure $d^n x\sqrt{G}/g_{YM}^2$ are background independent individually, which is what Seiberg and Witten showed in ref. [1].

3. Λ-symmetry in the noncommutative DBI theory

In section 2 we have analyzed the Λ-symmetry of noncommutative Yang-Mills theory, now we turn to the noncommutative DBI theory. The open string metric and the noncommutative parameter θ are related to the closed-string metric g and the constant background field B by [1]

$$G = g - Bg^{-1}B\,, \qquad \theta = -\frac{1}{g+B}B\frac{1}{g-B}\,. \tag{3.1}$$

The explicit expression for the mapping between noncommutative gauge fields and ordinary ones is valid only for small θ, so we have to restrict our discussion to this situation. From (3.1) we see that there are two possibilities to get a small value for θ: Either B is very large compared to g, or it is much smaller than g. In the case of large B, as argued in [1], the double scaling limit should be imposed to make the model have proper sense, then the corresponding noncommutative DBI theory is reduced to the noncommutative Yang-Mills theory. Hence we only consider the noncommutative DBI theory for small B. Since at small B

$$\theta = -g^{-1}Bg^{-1} + \left(g^{-1}B\right)^3 g^{-1} + \mathcal{O}(B^4) \tag{3.2}$$

and the B^2 term is absent, we will see below that it is good enough to use $\theta = -g^{-1}Bg^{-1}$ if we calculate to subleading order. Then we have

$$B = -g\theta g\,, \qquad G = g - g\theta g\theta g\,, \qquad G_s = g_s \det{}^{1/2}\left(1 + g^{-1}B\right) \tag{3.3}$$

and their variations under a Λ-transformation are

$$\begin{aligned} \delta B &= -g\delta\theta g\,, \qquad \delta G = -\left(g\delta\theta g\theta g + g\theta g\delta\theta g\right) \\ \delta G_s &= -\frac{1}{2}g_s \det{}^{1/2}\left(1 + g^{-1}B\right) \mathrm{Tr}\left(\theta g\delta\theta g\right). \end{aligned} \tag{3.4}$$

Under a Λ-transformation $F \to F + g\delta\theta g$ and the noncommutative gauge field strength is

$$\begin{aligned} \hat{F}\left(\theta + \delta\theta\right) = {} & F + g\delta\theta g - \left(F + g\delta\theta g\right)\left(\theta + \delta\theta\right)\left(F + g\delta\theta g\right) - \\ & - \left[A_k - \frac{1}{2}\left(g\delta\theta g\right)_{ks} x^s\right]\left(\theta + \delta\theta\right)^{kl}\partial_l F + \mathcal{T}_{(\theta+\delta\theta)^2} + \\ & + \text{higher order terms}\,. \end{aligned} \tag{3.5}$$

The variation of $\hat{F}$ under a Λ-transformation is defined by

$$\begin{aligned} \delta\hat{F}\left(\theta\right) = \hat{F}\left(\theta + \delta\theta\right) - \hat{F}\left(\theta\right) = {} & g\delta\theta g - F\delta\theta F - A_k\delta\theta^{kl}\partial_l F - g\delta\theta g F - \\ & - F\theta g\delta\theta g + \frac{1}{2}\left(g\delta\theta g\right)_{ks} x^s\theta^{kl}\partial_l F + K_{\theta\delta\theta}\,, \end{aligned} \tag{3.6}$$

where the leading and subleading terms are the orders of $\delta\theta$, $\theta\delta\theta$ respectively, and $K_{\theta\delta\theta}$ represents the $\theta\delta\theta$ order terms coming from $\mathcal{T}_{(\theta+\delta\theta)^2} - \mathcal{T}_{\theta^2}$, which can be obtained by replacing one of the θ's by $\delta\theta$ in $\mathcal{T}_{\theta^2}$. It can be written as

$$\begin{aligned} K_{\theta\delta\theta} = & \; F\delta\theta F\theta F + F\theta F\delta\theta F + \frac{1}{2} A_k \delta\theta^{kl} \left(\partial_l A_m + F_{lm}\right) \theta^{mn} \partial_n F + \\ & + \frac{1}{2} A_k \theta^{kl} \left(\partial_l A_m + F_{lm}\right) \delta\theta^{mn} \partial_n F + A_k \delta\theta^{kl} \partial_l \left(F\delta F\right) + A_k \theta^{kl} \partial_l \left(F\delta\theta F\right) + \\ & + \delta\theta^{kl} \theta^{mn} A_k A_m \partial_l \partial_n F \, . \end{aligned} \tag{3.7}$$

Using eqs. (3.4)–(3.6), $\delta G + \delta\hat{F}$ can be expressed to subleading order as

$$\delta G + \delta\hat{F} = P_1\left(\delta\theta\right) + P_2\left(\theta\delta\theta\right), \tag{3.8}$$

where $P_1\left(\delta\theta\right)$ and $P_2\left(\theta\delta\theta\right)$ represent the terms with the orders of $\delta\theta$ and $\theta\delta\theta$. They are given by

$$P_1\left(\delta\theta\right) = g\delta\theta g - F\delta\theta F - A_k \delta\theta^{kl} \partial_l F \, , \tag{3.9}$$

$$\begin{aligned} P_2\left(\theta\delta\theta\right) = & -g\delta\theta g\theta\left(g+F\right) - \left(g+F\right)\theta g\delta\theta g + \frac{1}{2}\left(g\delta\theta g\right)_{ks} x^s \theta^{kl} \partial_l F + \\ & + F\delta\theta F\theta F + F\theta F\delta\theta F + \frac{1}{2} A_k \delta\theta^{kl} \left(\partial_l A_m + F_{lm}\right) \theta^{mn} \partial_n F + \\ & + \frac{1}{2} A_k \theta^{kl} \left(\partial_l A_m + F_{lm}\right) \delta\theta^{mn} \partial_n F + A_k \delta\theta^{kl} \partial_l \left(F\theta F\right) + \\ & + A_k \theta^{kl} \partial_l \left(F\delta\theta F\right) + \delta\theta^{kl} \theta^{mn} A_k A_m \partial_l \partial_n F \, . \end{aligned} \tag{3.10}$$

The matrix $\left(G+\hat{F}\right)^{-1}$ and the square root of the determinant of $G+\hat{F}$, written in terms of g and F to subleading order, are

$$\begin{aligned} \frac{1}{G+\hat{F}} & = \frac{1}{g+F}\left[1 + \left(F\theta F + \theta^{kl} A_k \partial_l F\right)\frac{1}{g+F}\right] \\ \det{}^{1/2}\left(G+\hat{F}\right) & = \det{}^{1/2}\left(g+F\right)\left\{1 - \frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{g+F}\left(F\theta F + A_k \theta^{kl} \partial_l F\right)\right]\right\}, \end{aligned} \tag{3.11}$$

where the θ^2 term in G is omitted.

The noncommutative DBI lagrangean is given by [1]

$$\hat{\mathcal{L}}_{DBI}\left(\hat{A}\right) = \frac{1}{G_s} \det{}^{1/2}\left(G+\hat{F}\right). \tag{3.12}$$

The variation of $\hat{\mathcal{L}}_{DBI}$ under a Λ-transformation is

$$\delta\hat{\mathcal{L}}_{DBI} = \frac{1}{2G_s} \det{}^{1/2}\left(G+\hat{F}\right)\left[-\frac{2\delta G_s}{G_s} + \mathrm{Tr}\,\frac{1}{G+\hat{F}}\left(\delta G + \delta\hat{F}\right)\right]. \tag{3.13}$$

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Inserting (3.3), (3.4) and (3.8)–(3.11) into (3.13), we have, to subleading order,

$$\begin{aligned}
\delta\hat{\mathcal{L}}_{DBI} &= \frac{1}{2G_s}\det{}^{1/2}(g+F)\left\{1-\frac{1}{2}\operatorname{Tr}\left[\left(F\theta F+\theta^{kl}A_k\partial_l F\right)\frac{1}{g+F}\right]\right\}\times\\
&\quad\times\operatorname{Tr}\left\{\theta g\delta\theta g+\frac{1}{g+F}P_1(\delta\theta)+\right.\\
&\qquad\left.+\frac{1}{g+F}\left[P_2(\theta\delta\theta)+\left(F\theta F+\theta^{kl}A_k\partial_l F\right)\frac{1}{g+F}P_1(\delta\theta)\right]\right\}+\\
&\quad+\text{higher order terms}\,,\\
&= \frac{1}{2G_s}\left(\delta\mathcal{L}_1+\delta\mathcal{L}_2\right)+\text{higher order terms}\,,
\end{aligned} \tag{3.14}$$

where $\delta\mathcal{L}_1$ is of order $\delta\theta$, and $\delta\mathcal{L}_2$ is of order $\theta\delta\theta$. We find

$$\delta\mathcal{L}_1 = \det{}^{1/2}(g+F)\operatorname{Tr}\frac{1}{g+F}P_1(\delta\theta)\,, \tag{3.15}$$

$$\begin{aligned}
\delta\mathcal{L}_2 &= \det{}^{1/2}(g+F)\left\{-\frac{1}{2}\left[\operatorname{Tr}\frac{1}{g+F}\left(F\theta F+\theta^{kl}A_k\partial_l F\right)\right]\left[\operatorname{Tr}\frac{1}{g+F}P_1(\delta\theta)\right]+\right.\\
&\quad\left.+\operatorname{Tr}\left(\theta g\delta\theta g+\frac{1}{g+F}\left[P_2(\theta\delta\theta)+\left(F\theta F+\theta^{kl}A_k\partial_l F\right)\frac{1}{g+F}P_1(\delta\theta)\right]\right)\right\}.
\end{aligned} \tag{3.16}$$

First we examine the leading order term $\delta\mathcal{L}_1$. Inserting (3.9) into (3.15) we get

$$\begin{aligned}
\delta\mathcal{L}_1 &= \det{}^{1/2}(g+F)\operatorname{Tr}\frac{1}{g+F}\left(g\delta\theta g-F\delta\theta F-A_k\delta\theta^{kl}\partial_l F\right)\\
&= \det{}^{1/2}(g+F)\operatorname{Tr}\left[\frac{1}{g+F}\left(g\delta\theta g-F\delta\theta F\right)+\delta\theta F\right]+\text{total derivative}\,,
\end{aligned} \tag{3.17}$$

where in the second step we used

$$\det{}^{1/2}(g+F)\operatorname{Tr}\frac{1}{g+F}\partial_l F = 2\partial_l\det{}^{1/2}(g+F) \tag{3.18}$$

and then integrated by parts. Eq. (3.17) can be further reduced to

$$\begin{aligned}
\delta\mathcal{L}_1 &= \det{}^{1/2}(g+F)\operatorname{Tr}(\delta\theta g)+\text{total derivative}\\
&= \text{total derivative}\,,
\end{aligned} \tag{3.19}$$

where we have exploited that $\delta\theta$ is an antisymmetric matrix and that g is symmetric, thus $\operatorname{Tr}(\delta\theta g)=0$.

Eq. (3.19) shows that under a Λ-transformation the leading order term of the variation of the noncommutative DBI lagrangean is a total derivative.

Next we consider the subleading order term $\delta\mathcal{L}_2$. Plugging (3.9) and (3.10) into (3.16) we get

$$
\begin{aligned}
\delta\mathcal{L}_2 = {} & \det{}^{1/2}(g+F)\times \\
& \times\Big\{-\frac{1}{2}\left[\mathrm{Tr}\,\frac{1}{g+F}(g\delta\theta g - F\delta\theta F - A_k\delta\theta^{kl}\partial_l F)\right]\left[\mathrm{Tr}(F\theta F+\theta^{kl}A_k\partial_l F)\frac{1}{g+F}\right]+ \\
& + \mathrm{Tr}\Big(\theta g\delta\theta g + \frac{1}{g+F}\Big[-g\delta\theta g\theta(g+F)-(g+F)\theta g\delta\theta g + \\
& + \frac{1}{2}(g\delta\theta g)_{ks}\,x^s\theta^{kl}\partial_l F + F\delta\theta F\theta F + F\theta F\delta\theta F + \\
& + \frac{1}{2}A_k\delta\theta^{kl}(\partial_l A_m + F_{lm})\theta^{mn}\partial_n F + \frac{1}{2}A_k\theta^{kl}(\partial_l A_m + F_{lm})\delta\theta^{mn}\partial_n F + \\
& + A_k\delta\theta^{kl}\partial_l(F\theta F) + A_k\theta^{kl}\partial_l(F\delta\theta F) + \delta\theta^{kl}\theta^{mn}A_kA_m\partial_l\partial_n F\Big] + \\
& + \frac{1}{g+F}(F\theta F+\theta^{kl}A_k\partial_l F)\frac{1}{g+F}(g\delta\theta g - F\delta\theta F - A_m\delta\theta^{mn}\partial_n F)\Big)\Big\} \qquad (3.20)
\end{aligned}
$$

$\delta\mathcal{L}_2$ seems quite complicated. The strategy that we will use to simplify $\delta\mathcal{L}_2$ is to exploit (3.18) and the equation

$$
\begin{aligned}
2\partial_l\partial_n\det{}^{1/2}(g+F) = {} & \det{}^{1/2}(g+F)\left[\frac{1}{2}\left(\mathrm{Tr}\,\frac{1}{g+F}\partial_l F\right)\left(\mathrm{Tr}\,\frac{1}{g+F}\partial_n F\right) - \right. \\
& \left. - \mathrm{Tr}\left(\frac{1}{g+F}\partial_l F\frac{1}{g+F}\partial_n F\right) + \mathrm{Tr}\,\frac{1}{g+F}\partial_l\partial_n F\right], \qquad (3.21)
\end{aligned}
$$

and then integrate by parts, keeping terms like $\partial_l\,(F\theta F)$ unchanged. Then we have

$$
\begin{aligned}
\delta\mathcal{L}_2 = {} & \det{}^{1/2}(g+F)\times \\
& \times\Big\{-\frac{1}{2}\left[\mathrm{Tr}\,\frac{1}{g+F}(g\delta\theta g - F\delta\theta F)\right](\mathrm{Tr}\,\frac{1}{g+F}F\theta F) + \\
& + \partial_l\left[A_k\theta^{kl}\,\mathrm{Tr}\,\frac{1}{g+F}(g\delta\theta g - F\delta\theta F)\right] - \partial_l(A_k\delta\theta^{kl}\,\mathrm{Tr}\,\frac{1}{g+F}F\theta F) + \\
& + 2\partial_l\partial_n(A_kA_m\delta\theta^{kl}\theta^{mn}) - \mathrm{Tr}(\theta g\delta\theta g) - (g\delta\theta g)_{ks}(\partial_l\,x^s)\theta^{kl} + \\
& + \mathrm{Tr}\,\frac{1}{g+F}[F\delta\theta F\theta F + F\theta F\delta\theta F + A_k\delta\theta^{kl}\partial_l(F\theta F) + A_k\theta^{kl}\partial_l(F\delta\theta F)] + \\
& - \delta\theta^{kl}\theta^{mn}\partial_n[A_k(2\partial_l A_m - \partial_m A_l)] - \theta^{kl}\delta\theta^{mn}\partial_n[A_k(2\partial_l A_m - \partial_m A_l)] + \\
& + \mathrm{Tr}\,\frac{1}{g+F}F\theta F\frac{1}{g+F}(g\delta\theta g - F\delta\theta F) - \mathrm{Tr}\,\frac{1}{g+F}F\theta F\frac{1}{g+F}A_k\delta\theta^{kl}\partial_l F + \\
& + \mathrm{Tr}\,\frac{1}{g+F}\theta^{kl}A_k\partial_l F\frac{1}{g+F}(g\delta\theta g - F\delta\theta F)\Big\} + \text{total derivative}. \qquad (3.22)
\end{aligned}
$$

After a straightforward calculation with many cancellations we are left with

$$
\begin{aligned}
\delta\mathcal{L}_2 = \mathrm{Tr}\Big[& -F\delta\theta F\theta + \frac{1}{g+F}(F\delta\theta F\theta F + F\theta F\delta\theta F) + \\
& + \frac{1}{g+F}F\theta F\frac{1}{g+F}(g\delta\theta g - F\delta\theta F)\Big] + \text{total derivative}. \qquad (3.23)
\end{aligned}
$$

If we rewrite $g\delta\theta g - F\delta\theta F$ as

$$g\delta\theta g - F\delta\theta F = -\left(g+F\right)\delta\theta\left(g+F\right) + \left(g+F\right)\delta\theta g + g\delta\theta\left(g+F\right) \tag{3.24}$$

we can further simplify (3.23) and obtain

$$\delta\mathcal{L}_2 = \text{total derivative}\,, \tag{3.25}$$

which shows that the subleading order term of the variation of the noncommutative DBI lagrangean is a total derivative under Λ-transformations.

From the above calculation we have seen that the check of the Λ-symmetry for the noncommutative DBI action is high nontrivial, and we believe that if the ordinary gauge theory has Λ-symmetry, the corresponding noncommutative gauge theory should also possess this symmetry.

4. Summary and discussions

So far we have examined the Λ-symmetry of noncommutative Yang-Mills theory on $\mathbb{R}^n$. We have calculated the variation of the noncommutative Yang-Mills lagrangean including the subleading order terms. We have found that the leading and subleading order terms in the variation of the lagrangean are total derivatives, which indicates that the noncommutative Yang-Mills action is background independent if the ordinary gauge field damps fast enough at infinity. In the context of noncommutative Yang Mills theory on ordinary spacetime, we have shown that the quantity Q defined in [1] is not background independent for a general gauge field A. Only if the field strength F takes the constant value, the quantity Q is background independent. For the noncommutative DBI theory we have found that for small B the variation of the noncommutative DBI lagrangean under a Λ-transformation is a total derivative to subleading order. Thus, when the ordinary gauge field damps fast enough at the infinity, the noncommutative DBI action possesses Λ-symmetry, which is different from the ordinary DBI theory, where the lagrangean is manifestly Λ-symmetric. Since our check for Λ-symmetry of noncommutative gauge theories, which act as the low energy effective theories of D-branes, is highly non-trivial, we conclude that if the ordinary gauge theory has Λ-symmetry, its noncommutative counterpart should also possess this symmetry.

In the present paper, the Λ-transformation is realized in the framework of ordinary gauge fields, that is, we express $\delta\hat{F}$ in terms of A, F and θ, so that our discussion has to be restricted to small values of θ. It would be interesting to see whether it is possible to discuss Λ-transformations on noncommutative gauge field directly. In [6] it has been shown that Λ-symmetry could be modified in the presence of the $B-$field, similarly to the λ-symmetry. Under this deformed Λ-transformation, the B-field transforms as: $B \to B + d\Lambda + i\{\Lambda, A\}_{MB}$. It would be interesting to study the relation between the deformed Λ-transformation and the present one explicitely.

Recently, a hybrid point splitting regularization that leads to the Seiberg-Witten description including the general two-form Φ was found [7]. This suggests an investigation about how the Λ-transformation works in the presence of the general two-form Φ, since in this case we have more freedom for the description [1], which now can be realized within the standard renormalization scheme by some freedom in changing variables. We hope to answer these questions in the near future.

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